Formative Conversation Starters: Math

GRADE 8

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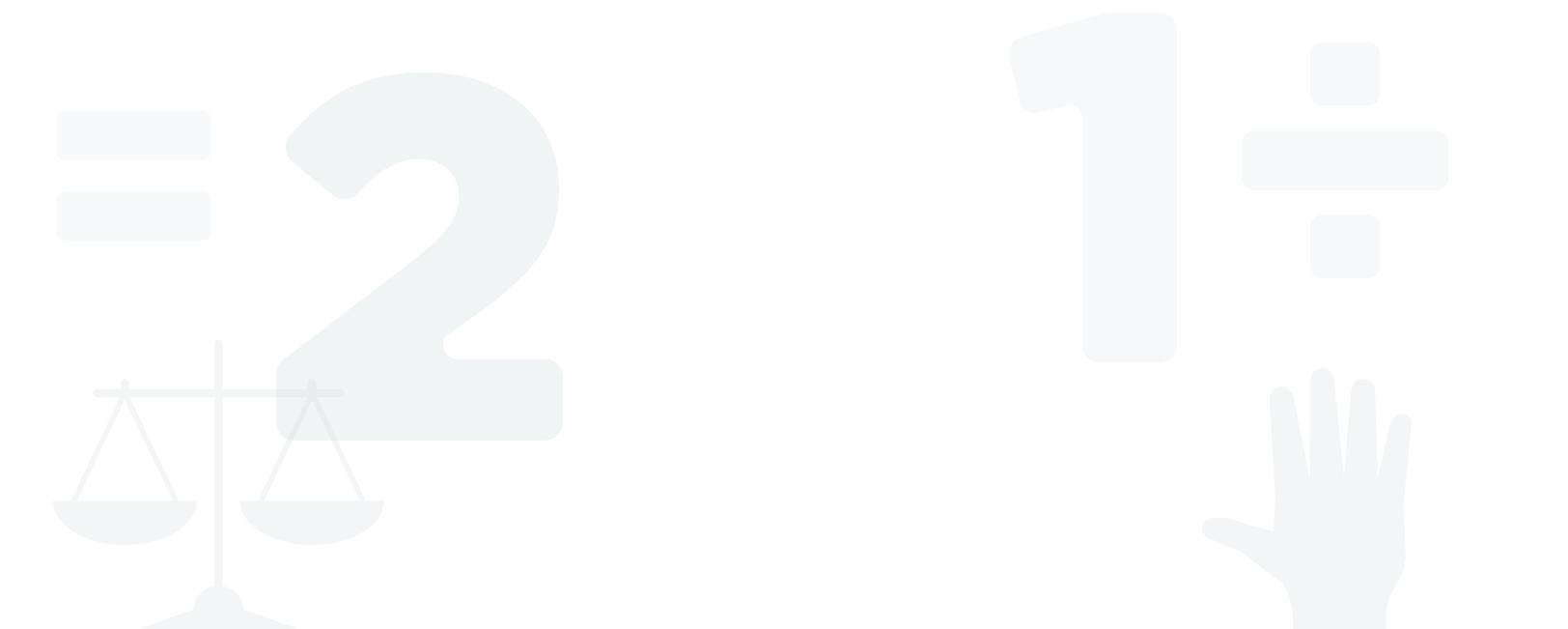
Formative Conversation Starters

Student understanding is more about growing than it is about getting. As educators, we might speak of some students "getting it" and other students not. This conclusion, though, is not fair to students. A student who may not seem to "get it" does understand *something*, even if that *something* may not yet have grown into a robust web of thinking. To reach students' full knowledge, we can use guided conversations: Formative Conversation Starters.

Purpose: The purpose of Formative Conversation Starters is to help teachers reveal student understanding about key ideas in mathematics and to identify their students' ways of thinking.

Audience: The intended audience is teachers, who will use these questioning strategies with students.

Application: Teachers may wish to use these conversation starters in one-on-one conferences with students or in small groups.

Formative Conversation Starters approach student knowledge by presenting a single standards-based assessment item and leveraging the item to elicit conversation through clustered questioning. The goal of this activity is not to tell students what to think, but to help teachers better uncover how students are currently thinking about mathematical concepts. The conversations provide opportunities for students to communicate how they are thinking about mathematics.

Mathematical Ideas (BINSS – Big Ideas to Nurture Standards Sense-making)

As you read through these items, you will notice that we draw attention to a few specific mathematical ideas. These ideas correspond to important ways of thinking that all students should develop and continue to refine. They include:

- **Operations:** Students begin to develop meanings for operations in kindergarten (e.g., addition is putting together). As they progress, the numbers involved—and operational meanings—extend. Students should develop ways of thinking that enable them to connect operation meanings to everyday use of those operations. Operations should never be disconnected from meaning. Division of fractions, for example, is still a form of division and should connect to a meaning of division.

- **Place Value:** Knowledge of place value is essential, and students should develop ways of thinking about place value that enable them to see the relationships between places. For example, they can think of a value in one place as 10 times that same value in the place to the right (*or a bundle of 10*), and they can carry that thinking between any places in any direction. They should be able to use that understanding effortlessly to compose and decompose quantities and to connect place-value understanding to operations.

- **Comparisons:** Comparisons can be either additive or multiplicative, with context guiding which is most appropriate. A multiplicative comparison is relative, describing one quantity in terms of another (e.g., 6 meters is <u>3 times as large</u> as 2 meters). Additive comparisons are absolute; the comparison is based on some other quantity (e.g., 6 meters is <u>4 meters more</u> than 2 meters). Students should have ways of thinking that help them determine which comparison to use or how an existing comparison is additive or multiplicative.

- **Measurement:** Geometric measurement is ultimately understood as the result of a multiplicative comparison between common attributes of two measurable quantities, and the result describes how many copies of a are contained in b. Equivalently, measurement addresses a times-as-large comparison such as "a is n times as large as b." Students thinking about measurement should have a clear understanding of which attribute is being measured and the comparison of two objects with that attribute, where one object's attribute is measured in terms of the other.

- **Fractions:** A fraction is a single number. It is a number just as 1, or 100, or 37,549 are, and it has a location on the number line. Students should be able to think of a fraction as a number and treat it as such. The fraction a/b can be thought of

as a copies of $1/b$, where $1/b$ is the length of a single part when the interval from 0 to 1 is partitioned into b parts. Two fractions are equivalent when they share a location on the number line.

- **Formulas:** Mastery of formulas (and procedures) is not the goal. Formulas are not the mathematics; they should be seen as shortcuts to help accomplish something with the mathematics. Students should have ways of thinking about the formulas that enable them to make sense of the quantities and to determine why quantities are connected with the indicated operations. Students should have mental and mathematical ways to reinvent useful formulas (e.g., $A=\pi r^2$ means three-and-a-bit copies of the square with area r^2).

- **Variables:** Students should have ways of thinking that enable them to distinguish between unknowns that vary and unknowns that represent some fixed value. For example, in the equation $6=3x+2$, x represents some unknown fixed value that makes the equation true. In $y=3x+2$, y and x vary with each other. In $y=mx+b$, y and x vary with each other, while m and b are typically nonvarying constants (parameters) within a problem.

- **Covarying Quantities:** Single quantities can vary, but students also need to consider situations where two quantities vary together. For example, in the equation $y=3x+2$, as the quantity x varies, the quantity y varies. It can be helpful to think about the relationship between covarying quantities in terms of how changes in x result in changes in y (e.g., as x increases by 1, y changes by . . .).

- **Proportional Relationships:** Proportional relationships require two covarying quantities. Those quantities must be measurable in some way, and the measures of those quantities scale in tandem. When one quantity changes by a scale factor, the other quantity also changes by the same scale factor. For example, doubling one quantity's value results in a doubling of the other quantity's value. Students should have ways of thinking that allow them to distinguish the two varying quantities in any proportional relationship and to explain how the quantities change by the same scale factor. It is important to note that proportional relationships are not synonymous with proportions. See this whitepaper for more information.

- **The Equal Sign:** The equal sign works in multiple ways in mathematics, even though the symbol does not change. Students should be able to think about the symbol as being relational (e.g., 2 + 4 = 5 + 1) and as being operational (e.g., the output of a computation), and should be able to determine which role the symbol is playing based on the situation. Keep an eye out for when students may put together equal signs. If this happens, prompting the student with questions about what the equal sign means may help. For example, suppose a student writes: 9 x 8 = 72 = 126 – 72 = 54 (*in this case, the student might be using the equal sign to say "next, I will…"*).

In the Formative Conversation Starters, you will notice that we call out mathematical ideas and the ways of thinking associated with them when it makes sense to do so. We also group discussion prompts to focus the conversations. Some groupings target core understandings that underpin the content. Other groupings elicit flexibility of thinking or extend beyond the assessment item being discussed. All of these methods are intended to provide opportunities for the teacher to listen to students and to reflect on how students might be thinking about mathematics in the standards-based assessment item.

The progressive question-and-answer strategy can be used to elicit evidence of students' ways of thinking about a topic or concept, with the purpose of guiding instruction.

How to Conduct a Formative Conversation

1. The questions for each item were developed to help teachers elicit information about students' ways of thinking about the content in the item and about mathematical ideas. These questions are suggestions, however, and not intended to be used as a script. The conversations teachers have will vary by student. While the questions for an item are laid out in a progression, teachers should vary the order to adapt to students' responses. Teachers should also keep in mind that students' responses may point to ways of thinking that are not addressed by the provided questions. In these cases, teachers should pursue those student understandings with their own line of questioning. There are several actions teachers can take to prepare for formative conversations:

    a. Become very familiar with the task and the questions ahead of time. This ensures that teachers can select the most appropriate next question based on how the students are responding.

    b. Provide students with tools to help them answer the questions. Depending on the task, these tools might include a manipulative, drawing paper, graph paper, or individual whiteboards and markers.

    c. Have a list of questions to help further probe what students are thinking. Some examples are:

        i. Can you tell me more about that?

        ii. You look like you're really thinking about this. What are you thinking?

        iii. Can you draw me a picture/write an equation?

        iv. How did you get that answer?

        v. Is there another way that you could find that answer?

    d. Make a plan to track what students say during the conversation: record the conversation, take notes, or have an observer take notes.

2. Teachers should ask questions without judgment. Student responses should not be labeled as right or wrong, and follow-up questions should be asked regardless of whether students give a correct response. Teachers should avoid commenting on students' responses other than to ask follow-up questions or to clarify what a student has said. Other students, however, should be encouraged to agree or disagree in a small group setting.

3. One of the most important parts of the formative conversation is what comes after the conversation: how will a teacher use the information about student thinking when planning instruction. Consider these suggestions for how to act on a formative conversation:

    a. Identify the different mathematical ideas addressed in the formative conversation. Where did students make connections between the ideas? Where do the connections need to be strengthened?

    b. Identify what students already understand in order to build instruction on that understanding.

    c. Identify areas where students can deepen what they already understand.

    d. Identify ways that students are comfortable with expressing mathematical ideas, and plan how to expand their capabilities. How were students most comfortable expressing or explaining what they understand? Were they more inclined to create a graphical representation of their thinking? Did they prefer to explain verbally? Did they use equations, or did they prefer to use graphs?

4. Some Conversation Starters include extension questions. These questions are provided as a way to elicit thinking beyond the grade-level of the BINSS.

8.1 Covarying Quantities

This activity focuses on student thinking about the relationship between covarying quantities in terms of how changes in x result in changes in y.

Covarying Quantities: Single quantities can vary, but students also need to consider situations where two quantities vary together. For example, in the equation $y = 3x + 2$, as the quantity x varies, the quantity y varies. It can be helpful to think about the relationship between covarying quantities in terms of how changes in x result in changes in y (e.g., as x increases by 1, y changes by . . .).
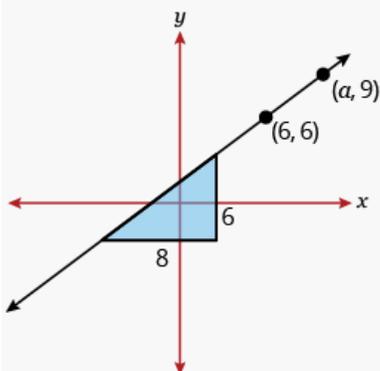
ITEM ALIGNMENT	*This item focuses on slope. However, it also provides an opportunity to talk about similar triangles, unit rates, proportional relationships, and linear equations.*
CCSS: 8.EE.B	

THE CONVERSATION STARTER

This question has two parts. Use the graph to answer Part A and Part B.

Part A
What is the slope of the line? Enter the answer in the box.

Part B
What is the value of a? Enter the answer in the box.

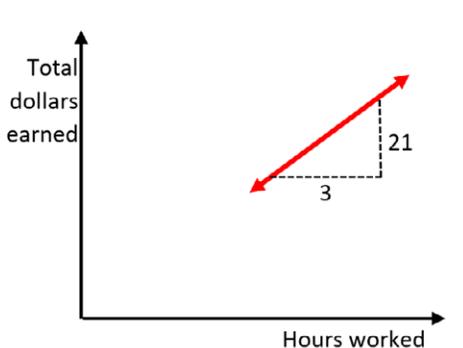# CONVERSATION PATHS (QUESTIONS ONLY)*

A. Problem Solving: Orientation

What is this question asking?

- What makes the question challenging?

B. Content: Proportional Relationships (Connections to Unit Rate & Slope)

Look at the information given in this graph. What can you tell me about the situation?

- Would you expect the line to pass through the origin? Why or why not?

  - What real-life situation could prevent the line from going through the origin?

Look at this graph. Does this graph show a relationship where x is proportional to y?
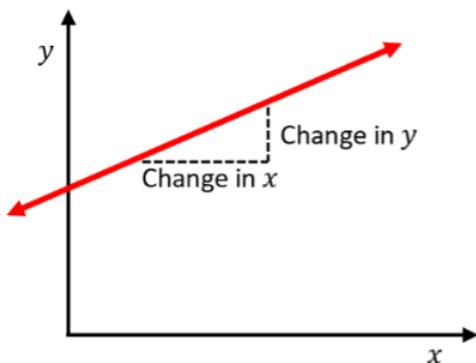
- In this next graph, is the *change* in x proportional to the change in y? Explain how you know.

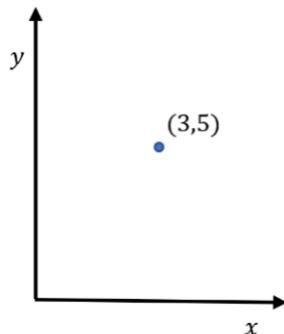- How is slope related to similar triangles?

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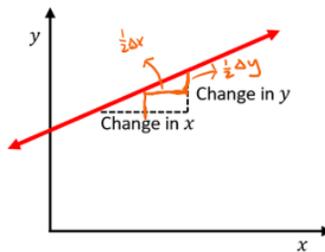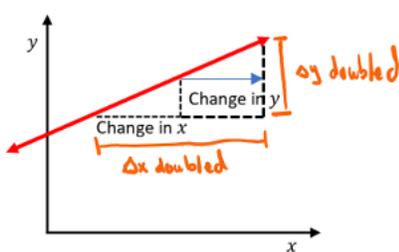
C. Content: Slope (Meaning)

Suppose you find out the slope of a line is -³⁄₇. What does that mean?

- If x increases by 7 units, what will happen to y?

- If x increases by 1, what will happen to y?

- If x increases by 2, what will happen to y?

- If x decreases by 1, what will happen to y?

## D. Content: Linear Equations (Meaning)

Look at this graph.

- If this point is on a line with a slope of –2, what is y when x is 2? Explain.

- What is y when x is 0? Explain.

- What is an equation for this line?

- How can you find the equation of any line when you have the slope and any point?

E. Content: Linear Graphs (Interpretation)

Imagine the graph in the original item represents how much a tree grew, in feet, based on the number of years since it was planted in a park. What does the point (6, 6) tell you?

- What does the point $(a,9)$ tell you?

- What does the vertical intercept tell you?
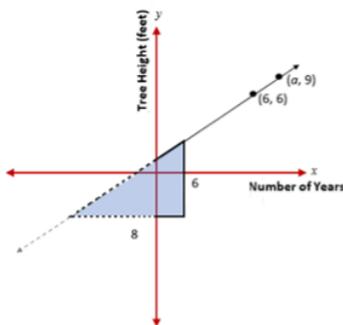
- What can you tell me about the triangle on the graph? What do you know about it?

  - What do the 6 and the 8 on the triangle mean?

  - How might you use the 6 and the 8 in this situation (tree height and number of years)?

  - Why does it make more sense to compare the 6 and 8 multiplicatively rather than additively?

  - If you know the tree grew at ¾ of a foot per year and is 6 feet tall after 6 years, how can you find the starting height?

  - If another point on the line is (14, b), what does b represent? Explain how you would find the value of b.

CONVERSATION PATHS (ANNOTATED)*

## A. Problem Solving: Orientation

What is this question asking?

> *The question is asking two things. First, it is asking to determine the slope of the given line. Also, it is asking for the missing x-coordinate for one point on the line.*

- What makes the question challenging?

> *For example, finding the slope itself may be challenging for someone struggling with the idea of slope. Some may initially see no way to approach the problem without first finding the equation of the line. Others might be challenged in thinking about how to tandemly scale a change in x with a corresponding change in y.*

B. Content: Proportional Relationships (Connections to Unit Rate & Slope)

Look at the information given in this graph. What can you tell me about the situation?
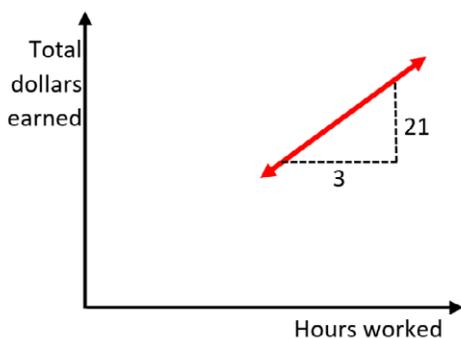
> *Students may observe several aspects of the situation given in this graph. The total dollars earned increases as the number of hours worked increases. When the number of hours increases by 3, the total number of dollars earned increases by $21. If 6 hours are worked, the total dollars earned should increase by $42. If the per-hour rate is constant, for each hour worked, the total dollars earned increases by $7. At this rate, a person working a standard 40-hour week will earn $280.*

- Would you expect the line to pass through the origin? Why or why not?

> *Yes. We assume that if the number of hours worked is 0 hours, the total dollars earned is $0.*

- What real-life situation could prevent the line from going through the origin?

> *Suppose a worker, like a plumber, charges a flat fee to start the work—such as a trip charge—and then charges by the hour. This would result in the graph intersecting the vertical axis at a positive value. Once on the job, they earn a certain amount of money per hour worked.*

Look at this graph. Does this graph show a relationship where x is proportional to y?
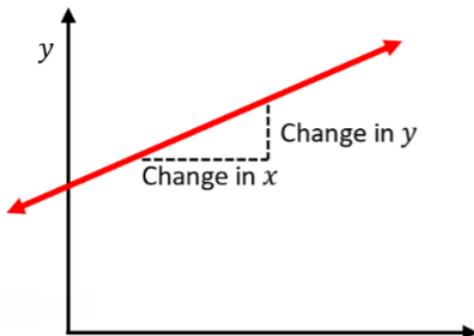
No, when measures of quantities scale in tandem, the line passes through the origin. Or, proportional relationships are modeled by equations of the form $y=kx$, and in this example there is a vertical intercept that is not (0, 0).

- In this next graph, is the *change* in x proportional to the change in y? Explain how you know.
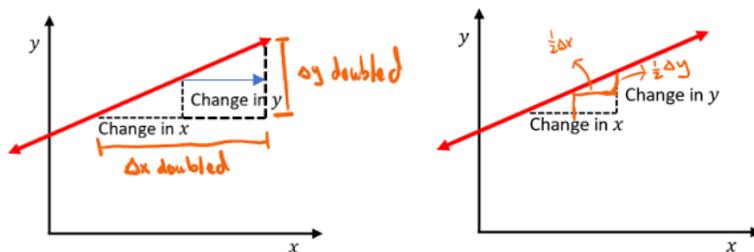
Yes. A student may explain that if the change in x was doubled, the change in y would be doubled. Or, if the change in x is cut in half, the change in y will be cut in half (see images).

- How is slope related to similar triangles?

Students might express the connection as demonstrated above.

C. Content: Slope (Meaning)

Suppose you find out the slope of a line is -⅗. What does that mean?

Do students see it more than just as a static "change in y over change in x"? For example, it means that the change in y will always be -⅗ times as large as a corresponding change in x. This could also be stated that each increase of 7 units in the input quantity will result in the output quantity decreasing by 3 units. Or, for every increase of 1 unit in the input quantity, the output quantity will decrease by ⅗ units.

- If x increases by 7 units, what will happen to y?

y will decrease by 3 units

- If x increases by 1, what will happen to y?

y will decrease by ⅗ units

- If x increases by 2, what will happen to y?

y will decrease by 2 copies of ⅗ units or ⁶⁄₇ units

- If x decreases by 1, what will happen to y?

y will increase by ⅗ units

D. Content: Linear Equations (Meaning)

Look at this graph.
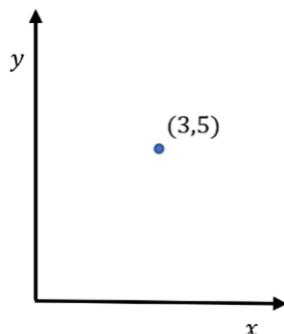
- If this point is on a line with a slope of –2, what is y when x is 2? Explain.

    With a slope of –2, if x increases 1 unit, then y will decrease by 2 units. Or, if x decreases by 1 unit, y will increase by 2 units. Therefore, the line passing through (3, 5) will also pass through (2, 7). Note that we decreased x by 1 unit, which increased y by 2 units (from $y = 5$ to $y = 7$).

- What is y when x is 0? Explain.

    When $x = 0$, $y = 11$. With a slope of –2, decreasing x by 2 units from $x = 2$ to $x = 0$ will create an increase in y of 4 units from $y = 7$ to $y = 11$.

- What is an equation for this line?

    $y = 11 - 2x$

- How can you find the equation of any line when you have the slope and any point?

    Are students able to think this through based on meanings rather than formulas? One strategy is to find the vertical intercept by leveraging the slope meaning and the known point. Then, we can express the equation of the line with slope a and vertical intercept $(0, b)$ as $y = ax + b$.

E. Content: Linear Graphs (Interpretation)

Imagine the graph in the original item represents how much a tree grew, in feet, based on the number of years since it was planted in a park. What does the point (6, 6) tell you?
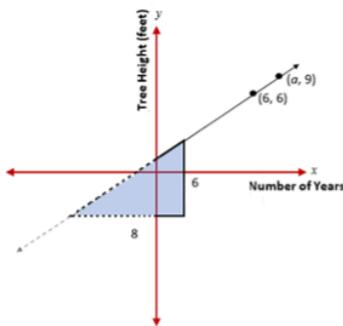
After 6 years, the tree will be 6 feet tall.

- What does the point $(a, 9)$ tell you?

    After 'a' years, the tree will be 9 feet tall.

- What does the vertical intercept tell you?

    The vertical intercept is the initial height of the tree when planted.

- What can you tell me about the triangle on the graph? What do you know about it?

    It's a right triangle. It has a height of 6 units and a base of 8 units, based on the way it is currently positioned.

    - What do the 6 and the 8 on the triangle mean?

        The 6 represents a change in output (tree height in feet) and the 8 represents a change in the input (number of years).

    - How might you use the 6 and the 8 in this situation (tree height and number of years)?

        We can compare these quantities multiplicatively to determine the ratio of the change in feet to the change in years, which is ⅝ or ¾. That is, there is growth of ⅝ foot per year. We could find the hypotenuse length, but it wouldn't be immediately useful in this context.

    - Why does it make more sense to compare the 6 and 8 multiplicatively rather than additively?

It wouldn't make sense to subtract feet and years. The resulting number would have no meaning in this context. Comparing 6 feet to 8 years multiplicatively (through division) reflects how the change in the number of feet is always ⅝ times as much as the change in number of years. In thinking about slope, this provides a measure of steepness that is true between any corresponding changes in years and growth.

- If you know the tree grew at ¾ of a foot per year and is 6 feet tall after 6 years, how can you find the starting height?

  Subtract 6 copies of ¾ from 6 feet. 6 − ¾ (6) = 1.5

- If another point on the line is (14, b), what does b represent? Explain how you would find the value of b.

  It would be the original height of 1.5 feet plus 14 years of growth at ¾ foot per year, or 1.5 + ¾ (14).

8.1 Shareables*

This question has two parts. Use the graph to answer Part A and Part B.
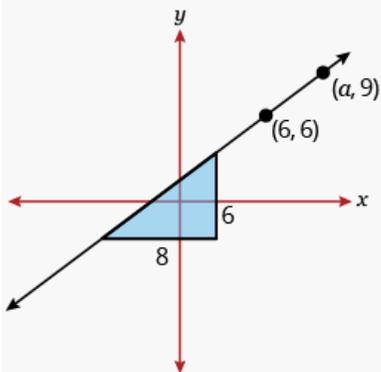
Part A
What is the slope of the line? Enter the answer in the box.

Part B
What is the value of a? Enter the answer in the box.

B.

D.

E.

8.2 The Equal Sign

This activity focuses on student thinking about solving problems in a real-world context using two linear equations in two variables.

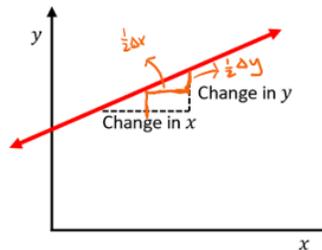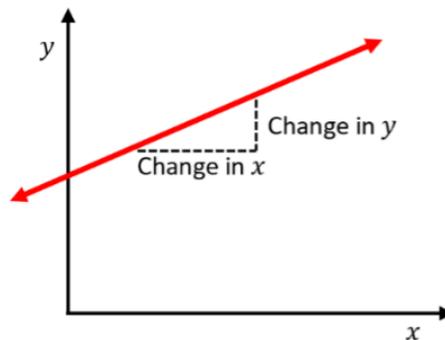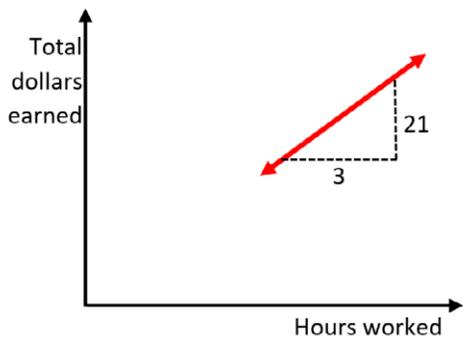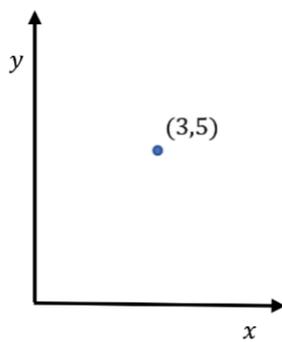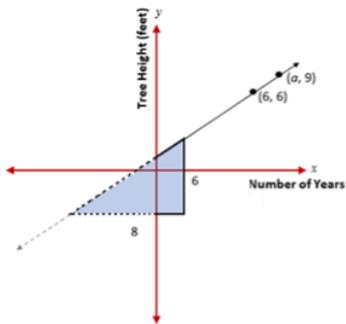**The Equal Sign:** The equal sign works in multiple ways in mathematics, even though the symbol does not change. Students should be able to think about the symbol as being relational (e.g., 2 + 4 = 5 + 1) and as being operational (i.e., the output of a computation), and should be able to determine which role the symbol is playing based on the situation. Keep an eye out for when students may put together equal signs. If this happens, prompting the student with questions about what the equal sign means may help. For example, suppose a student writes: 9 x 8 = 72 = 126 – 72 = 54 (*in this case, the student might be using the equal sign to say "next, I will…"*).

ITEM ALIGNMENT	This item focuses on solving systems of equations. However, it also provides an opportunity to talk about solving equations and meanings of terms and expressions.
CCSS 8.EE.C.8c	

THE CONVERSATION STARTER

Use the information to answer the question.

A school receives two shipments of desks and chairs. The shipping labels are shown below.

Shipment 1 of 2	Shipment 2 of 2
Item: Natural Wood Student Desk Quantity: 24	Item: Natural Wood Student Desk Quantity: 30
Item: Baby Blue Student Chair Quantity: 24	Item: Baby Blue Student Chair Quantity: 40
Total Shipment Weight: 744 pounds	Total Shipment Weight: 1020 pounds

A student is carrying the chairs in stacks of 3. How much does each stack weigh, in pounds? Enter the answer in the box.

[   ] pounds

CONVERSATION PATHS (QUESTIONS ONLY)*

A. Problem Solving: Orientation

What is this problem asking you to find?

- What information do you have?

- What information do you need to find?

- Do you need the information on the second shipment, or can you solve only with the information on the first shipment? Why or why not?

- How would you begin to answer this question? Would you use a drawing? Objects? An equation? A graph?

B. Content: Equations (Comparison)

How are these equations alike? How are they different?

$$3x + 5(2) = 15 \quad 3x + 5y = 15$$

- How would you "solve" the first equation?

- How would you "solve" the second equation?

What do you think of when you see equations like this?

$$\begin{cases} 3x + 5y = 10 \\ 2x - 3y = 15 \end{cases}$$

- How many solutions does the first equation have?

- How many solutions does the second equation have?

- How many solutions do they have in common? How can you be sure?

- What methods could be used to find the solution these two equations have in common?

In Shipment 2, there are 30 desks and 40 chairs with a combined weight of 1020 lbs. Suppose someone writes this equation:

$$30d + 40c = 1020$$

- What would be the units of the term $30d$? Why?

- What would the term $30d$ represent?

- What does d represent?

Suppose you have a situation where you know that 40 identical math books and 20 identical science books cost a total of $2,200.

- Can you determine the price of one math book? Why or why not?

- What other information could be helpful?

- After quickly reading the sentence, someone writes the equation $40m+20s=2,200$. What are the units of the term $20s$?

- What does the term $20s$ represent?

- What does the s represent?

- What does it mean to "solve" the equation $40m+20s=2{,}200$?

- How many solutions do you think you could find? Can you find at least one solution?

CONVERSATION PATHS (ANNOTATED)*

A. Problem Solving: Orientation

What is this problem asking you to find?

> ***The weight in pounds of a stack of 3 of the chairs.***

- What information do you have?

> ***The number of desks and the number of chairs and the total weight for each of two shipments.***

- What information do you need to find?

> ***The weight of one chair, which will allow us to find the weight of a stack of 3 chairs.***

- Do you need the information on the second shipment, or can you solve only with the information on the first shipment? Why or why not?

> ***If we only had the information for one shipment, there could be many possible desk and chair weights to obtain the total shipment weight given in the problem. For example, if only shipment 1 was provided, we can show that a desk weight of 18 pounds and a chair weight of 13 pounds will create a total shipment weight of 744 pounds. Also, a desk weight of 24 pounds and a chair weight of 7 pounds will create a total shipment weight of 744 pounds. But, with two shipment details provided, we can determine the exact weight of one desk and one chair. Only one combination of weights will produce the two total shipment weights given.***

- How would you begin to answer this question? Would you use a drawing? Objects? An equation? A graph?

> ***Consider pursuing whichever option students are considering.***

B. Content: Equations (Comparison)

How are these equations alike? How are they different?

$$3x + 5(2) = 15 \qquad 3x + 5y = 15$$

> ***They are alike in that both equations show that the expressions on the left side of the equations are both equal to 15 and both contain the term, $3x$. The equation on the left is the special case for the one on the right when $y = 2$. They are different in that in the first equation there is one unknown quantity, x. In the second equation there are two unknown quantities, x and y. In the single variable equation, there is a single value of x that makes the equation true. In the two-variable equation, there are many values for x and y that make the equation true.***

- How would you "solve" the first equation?

> ***One possibility: Think 5 x 2 = 10. Then think, 5 + 10 = 15. Therefore, since $3x + 10 = 15$, it follows that $3x = 5$. This leads to the solution, $x = \frac{5}{3}$.***

- How would you "solve" the second equation?

> ***We would need to find all pairs of values that make the equation true. Find all values of y such that $y = \frac{(15-x)}{5}$ or $y = 3 - (\frac{3}{5})x$. That is, all ordered pairs $(x, 3 - \frac{3}{5}x)$ show solutions to the equation. A graph is an ideal way to represent all of those solutions.***

$$\begin{cases} 3x + 5y = 10 \\ 2x - 3y = 15 \end{cases}$$

What do you think of when you see equations like this?

> *Do students see it simply as a system to solve using a procedure? Do they see it as two equations, each of which has an infinite number of solutions? Do they understand that those infinite solutions may match up for a particular (x, y) pair? Do they consider that there may be no shared solutions or that they might have the same solutions sets?*

- How many solutions does the first equation have?

  Infinite. The easiest way to represent that is with a line, where each point on the line represents a solution.

- How many solutions does the second equation have?

  As above, there are an infinite number of solutions.

- How many solutions do they have in common? How can you be sure?

  They have one solution in common. Since they do not represent the same line and the lines do not have the same slope, they will intersect at one point. The point of intersection of the lines is the one common solution.

- What methods could be used to find the solution these two equations have in common?

  Solution methods may include graphing each line, or solving systems of equations using the algebraic procedures of substitution or elimination.

In Shipment 2, there are 30 desks and 40 chairs with a combined weight of 1020 lbs. Suppose someone writes this equation:

$$30d + 40c = 1020$$

- What would be the units of the term $30d$? Why?

  Pounds. The 1,020 on the right side represents pounds. This means that the left side of the equation must also represent 1,020 pounds. As it is the result of two expressions added together, each of those must represent pounds as well.

- What would the term $30d$ represent?

  The total number of pounds of the desks.

- What does d represent?

  Since 30 is the number of desks, d would represent the weight of one desk.

Suppose you have a situation where you know that 40 identical math books and 20 identical science books cost a total of $2,200.

- Can you determine the price of one math book? Why or why not?

  No. There are many possible answers that would satisfy this situation. For example, suppose a math book costs $20. Then, 40 math books would cost $800. This leaves $1,400 in total for the science books. Since there are 20 science books, each could cost $70 (20×$70 = $1,400). Or, suppose a math book costs $30. Then, 40 math books would cost $1,200. This leaves $1,000 for science books. Since there are 20 science books, each could cost $50 (20×$50 = $1,000). Many other options exist.

- What other information could be helpful?

  The cost of one of the types of books would help us find the cost of the other type of book. Or, knowing information about another set of these books would also help us to find the cost of each. Or, knowing the relationship between the cost of the two types of books (e.g., the science book costs two times that of the math book.)

- After quickly reading the sentence, someone writes the equation $40m+20s=2,200$. What are the units of the term $20s$?

  Dollars. Since the right side of the equation is dollars, the left side must be as well. Because the two terms are added, and only like things can be added, each must represent dollars.

- What does the term $20s$ represent?

  The total cost of 20 books that cost s dollars each.

- What does the s represent?

The cost of a single science book.

- What does it mean to "solve" the equation $40m + 20s = 2{,}200$?

    Solving the equation means to find all pairs of values that make the equation true.

- How many solutions do you think you could find? Can you find at least one solution?

    (30, 50) is one solution. (15, 20) is another.

8.2 Shareables*

Use the information to answer the question.

A school receives two shipments of desks and chairs. The shipping labels are shown below.

Shipment 1 of 2	Shipment 2 of 2
Item: Natural Wood Student Desk Quantity: 24	Item: Natural Wood Student Desk Quantity: 30
Item: Baby Blue Student Chair Quantity: 24	Item: Baby Blue Student Chair Quantity: 40
Total Shipment Weight: 744 pounds	Total Shipment Weight: 1020 pounds

A student is carrying the chairs in stacks of 3. How much does each stack weigh, in pounds? Enter the answer in the box.

pounds

B.

$$3x + 5(2) = 15 \quad 3x + 5y = 15$$

$$\begin{cases} 3x + 5y = 10 \\ 2x - 3y = 15 \end{cases}$$

$$30d + 40c = 1020$$

8.3 Covarying Quantities

This activity focuses on students comparing properties of related functions displayed in different formats.

Covarying Quantities: Single quantities can vary, but students also need to consider situations where two quantities vary together. For example, in the equation $y=3x+2$, as the quantity x varies, the quantity y varies. It can be helpful to think about the relationship between covarying quantities in terms of how changes in x result in changes in y (e.g., as x increases by 1, y changes by . . .).
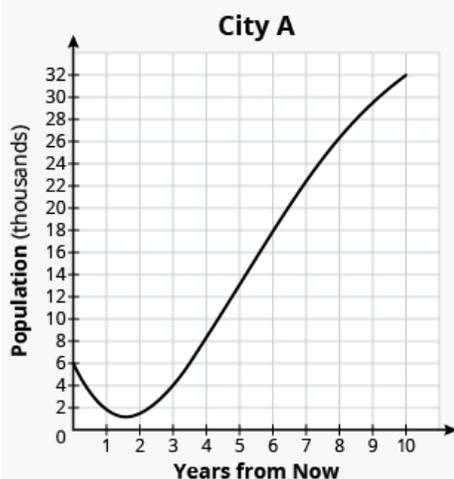
ITEM ALIGNMENT	*This item focuses on functions. However, it also provides an opportunity to talk about covarying quantities, linear growth, and meanings and representations of functions.*
CCSS 8.F.A.2	

THE CONVERSATION STARTER

Use the information to answer the question.

Two students set out to model the future populations of City A and City B. The first student graphs the future population of City A, in thousands, over the next 10 years. The graph of the model's function is shown.

City A

The second student guesses that City B's population is going to grow linearly and models the population in the table, as shown.

City B

Years from Now	1	3	5
Population (thousands)	11	15	19

How will the populations compare in 8 years? Select one choice from each set to complete the sentence.

Based on the models, the population of [**City A** / **City B**] should exceed [**City A** / **City B**] by approximately [1000 / 2000 / 3000 / 5000] people 8 years from now.

CONVERSATION PATHS (QUESTIONS ONLY)*

A. Problem Solving: Orientation

What is this problem asking you to do?

- What makes this task challenging?

B. Content: Function (Meaning)

What is a function?

- What are different ways you can represent functions?

- What advantages do different representations provide?

- Do functions always have equations?

- In this problem, what would be considered the input of the function? What would be the output?

- Could the following graphs represent city population functions? Why or why not?

C. Content: Linear Growth

Is the population of City A growing linearly?

- What does it mean when we say something "grows linearly"?

- What might it mean when we say something "grows exponentially"?

A third student created a table predicting the future population of City C.

Years from Now	2	3	5
Population (thousands)	17	21	29

- Does this population growth prediction appear linear?

- How does the population change each year?

- What do you think the population is now? Why?

- What should have been the population one year ago? How did you find it?

- Could you write an equation to model this data?

- Based on this, what will the population be in 20 years?

- When will the population reach 50,000?

D. Content: Functions (Interpretation, Graphs)

What does the graph titled "City A" tell you? What are some things that you can tell me about City A based on the graph?

The graph tells how City A's population is predicted to grow. It shows that the population will decrease and then it will increase. It tells me that in one year the population will decrease from 6,000 to 2,000 but then after three years, it will go up to 4,000. After 10 years, the population will have increased to 32,000 people.

- When will the population of City A be 3,000 people?

- **Extension:** Suppose this is the equation for the function shown in the graph for City A, where y is the population (in thousands) and x is the number of years from now.

$$y = \frac{1}{180}x^4 - \frac{19}{90}x^3 + \frac{439}{180}x^2 - \frac{187}{30}x + 6$$

- What is a solution to this equation?

- What is a solution to the following equation?

$$4 = \frac{1}{180}x^4 - \frac{19}{90}x^3 + \frac{439}{180}x^2 - \frac{187}{30}x + 6$$

E. Content: Functions (Interpretation, Tables)

What does the table titled "City B" tell you?

- How does City B's population change each year?

- If that is the case, what should City B's population be right now?

- What equation could we use to represent City B's population?

- Based on this model, how could you determine when City B's population will reach 100,000?

- Would your answer be a reliable estimate?

CONVERSATION PATHS (ANNOTATED)*

A. Problem Solving: Orientation

What is this problem asking you to do?

Compare a function represented in a graph to a function represented in a table. Compare City A's population growth with City B's population growth and make a prediction about which city will have more people.

- What makes this task challenging?

For example, some may find it difficult to compare the two functions because they are in different forms (a graph and a table). Others may find it challenging because one is a linear function and the other is non-linear. Some may find the amount of information to be challenging or they may have no immediate way to make sense of the problem.

B. Content: Function (Meaning)

What is a function?

A relationship (or rule) between two quantities such that as the input quantity x varies, it corresponds with exactly one output quantity, y.

- What are different ways you can represent functions?

Functions may be represented graphically, in a table, with an equation, verbally, or with some other set of rules.

- What advantages do different representations provide?

Graphs, for example, provide a visual that makes it easy to spot trends and important points. Tables might be useful to quickly identify patterns and specific values. Equations might be useful for calculation and algebraic manipulation.

- Do functions always have equations?

No, an equation is not a requirement for a function. For example, a function could just be a set of rules that describe an output based on an input.

- In this problem, what would be considered the input of the function? What would be the output?

The input is the number of years from now and the output is the population, in thousands, in that year.

- Could the following graphs represent city population functions? Why or why not?

Imagine dragging your right index finger from left to right along the years axis (the x-axis). For each point along that axis, there can only be one population. The first graph could be drawn in this way: As your finger moves across the x-axis, you can see a single population value at each point in time. The second and third graphs would require multiple populations at some times, or perhaps for your axis-tracing finger to go back in time, neither of which would make any sense.

C. Content: Linear Growth

Is the population of City A growing linearly?

No, not for the whole graph. But it does appear to grow fairly linearly from years 4 to 7.

- What does it mean when we say something "grows linearly"?

  It means that for each like-increment in x (such as a unit increase), y changes by a like amount each time. For example, if each increase in one year results in a population increase of 1,000, then the relationship between years and population is linear.

- What might it mean when we say something "grows exponentially"?

  It means that for each like-increment in x, y is multiplied by a like amount each time. For example, for every year that goes by, the population is multiplied by 2.

A third student created a table predicting the future population of City C.

Years from Now	2	3	5
Population (thousands)	17	21	29

- Does this population growth prediction appear linear?

  It could be linear. We see a constant rate of change of 4,000 people per year.

- How does the population change each year?

  The population increases by 4,000 people each year.

- What do you think the population is now? Why?

  We can go "back in time" two years from (2, 17) at a rate of 4,000 people per year. Therefore, we can justify that the population one year from now is 13,000 by subtracting 4,000 from 17,000, and the population right now, zero years from now, is 9,000 by subtracting 4,000 from 13,000.

- What should have been the population one year ago? How did you find it?

  5,000 people. 9,000 – 4,000 = 5,000

- Could you write an equation to model this data?

  Using P as the population (in thousands) and t as the number of years from now, $P = 9 + 4t$.

- Based on this, what will the population be in 20 years?

  $P = 9 + 4(20)$, so $P = 89$, making the population in 20 years 89,000 people.

- When will the population reach 50,000?

  Since $50 = 9 + 4t$, it follows that $t = 10.25$ years. So, after 10.25 years, the population will reach 50,000 given this constant rate of change of 4,000 people per year.

D. Content: Functions (Interpretation, Graphs)

What does the graph titled "City A" tell you? What are some things that you can tell me about City A based on the graph?

  The graph tells how City A's population is predicted to grow. It shows that the population will decrease and then it will increase. It tells me that in one year the population will decrease from 6,000 to 2,000, but then after three years, it will go up to 4,000. After 10 years, the population will have increased to 32,000 people.

- When will the population of City A be 3,000 people?

  The population of City A reaches 3,000 people in two different years. With the initial decrease in population, it reaches 3,000 people within the first year (approximately $t = 0.5$ years). Then, the population increases and reaches 3,000 people between years 2 and 3 (approximately $t = 2.75$ years).

- **Extension:** Suppose this is the equation for the function shown in the graph for City A, where y is the population (in thousands) and x is the number of years from now.

- What is a solution to this equation?

$$y = \frac{1}{180}x^4 - \frac{19}{90}x^3 + \frac{439}{180}x^2 - \frac{187}{30}x + 6$$

A solution would be any (x, y) pair that is on the graph.

- What is a solution to the following equation?

$$4 = \frac{1}{180}x^4 - \frac{19}{90}x^3 + \frac{439}{180}x^2 - \frac{187}{30}x + 6$$

A solution would be the value for x that creates an output value of 4. Looking at the graph, it looks like an answer would be around $x = 3$, or somewhere around $x = 0.4$. Note: A computer can tell us that both $x = 3$ and $x \approx 0.37376$ are solutions.

E. Content: Functions (Interpretation, Tables)

What does the table titled "City B" tell you?

The description says that City B's population is going to grow linearly. City B's population does not decrease and then increase. It only increases.

- How does City B's population change each year?

Based upon the table, the population increases by 4,000 people every 2 years. On average, this is 2,000 people per year.

- If that is the case, what should City B's population be right now?

Using the rate of 2,000 people per year, we can back up one year from (1, 11) and find that the population is 9,000 people right now (0, 9).

- What equation could we use to represent City B's population?

Using P to represent the population (in thousands) and t to represent the number of years from now, $P = 9 + 2t$.

- Based on this model, how could you determine when City B's population will reach 100,000?

We can work to find the number of years from now, t, such that $9 + 2t = 100$, where the 100 represents 100 thousand people. This means that $2t = 91$, and $t = 45.5$. The linear equation predicts that the population will be 100,000 in 45.5 years.
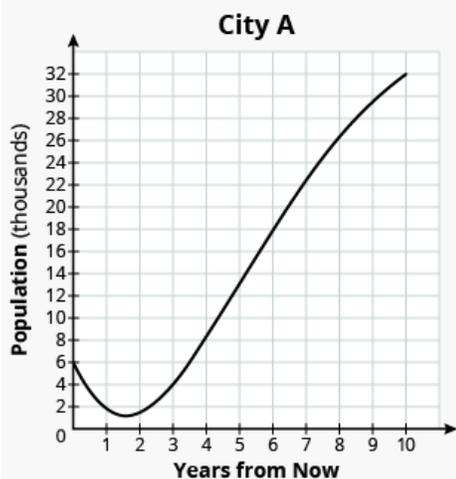
- Would your answer be a reliable estimate?

We wouldn't expect a city to continue to grow by the same amount every year for 45 years, so it would not be a very good model that far out.

8.3 Shareables*

Use the information to answer the question.

Two students set out to model the future populations of City A and City B. The first student graphs the future population of City A, in thousands, over the next 10 years. The graph of the model's function is shown.

City A

The second student guesses that City B's population is going to grow linearly and models the population in the table, as shown.

City B

Years from Now	1	3	5
Population (thousands)	11	15	19

How will the populations compare in 8 years? Select one choice from each set to complete the sentence.

Based on the models, the population of
[**City A** / **City B**] should exceed [**City A** / **City B**] by approximately [1000 / 2000 / 3000 / 5000] people 8 years from now.

B.

C.

Years from Now	2	3	5
Population (thousands)	17	21	29

D.

$$y = \frac{1}{180}x^4 - \frac{19}{90}x^3 + \frac{439}{180}x^2 - \frac{187}{30}x + 6$$

$$4 = \frac{1}{180}x^4 - \frac{19}{90}x^3 + \frac{439}{180}x^2 - \frac{187}{30}x + 6$$

8.4 Formulas

This activity focuses on student thinking about the Pythagorean Theorem.

Formulas: Mastery of formulas (and procedures) is not the goal. Formulas are not the mathematics; they should be seen as shortcuts to help accomplish something with the mathematics. Students should have ways of thinking about the formulas that enable them to make sense of the quantities and to determine why quantities are connected with the indicated operations. Students should have mental and mathematical ways to reinvent useful formulas (e.g., $A=\pi r^2$ means three-and-a-bit copies of the square with area r^2).

ITEM ALIGNMENT	
CCSS: 8.G.B	*This item focuses on the Pythagorean Theorem. However, it also provides an opportunity to talk about angles, square roots, right triangles, and equations.*

THE CONVERSATION STARTER

Use the information to answer the question.

A triangle ABC has side length $|AB| = 1$, $|BC| = \sqrt{2}$, and $|AC| = \sqrt{3}$.

What type of angle is each angle in the triangle? Choose a type for each angle.

Angle	Type		
$\angle ABC$	acute	obtuse	right
$\angle BCA$	acute	obtuse	right
$\angle CAB$	acute	obtuse	right

CONVERSATION PATHS (QUESTIONS ONLY)*

A. Problem Solving: Orientation

What is this question asking you?

- What makes it challenging?

- Without being precise on the lengths, could you sketch this triangle?

- Which angle would be $\angle ABC$ in your diagram?

B. Content: Pythagorean Theorem (Interpretation, Meaning)

Can line segments of length 1 cm, 2 cm, and 3 cm make a triangle?

- In order to make a triangle, what must be true about the three line segments?

- Can line segment lengths 1, √2, and √3 even make a triangle? How do you know?

What does this figure have to do with the Pythagorean theorem?
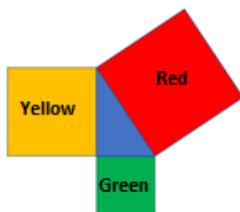
What happens if $c^2 > a^2 + b^2$? How can this diagram help you explain it?

What happens if $c^2 < a^2 + b^2$? How can this diagram help you explain it?

C. Content: Equations (Square roots)

In general, is $\sqrt{(a^2 + b^2)} = a + b$? Explain.

$$\sqrt{a^2 + b^2} = a + b$$

- Is $\sqrt{(a^2)} + \sqrt{(b^2)} = a + b$?

$$\sqrt{a^2} + \sqrt{b^2} = a + b$$

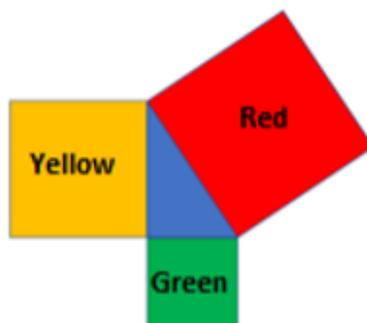- Is $\sqrt{(a^2)} + \sqrt{(b^2)} = |a| + |b|$?

$$\sqrt{a^2} + \sqrt{b^2} = |a| + |b|$$

D. Pythagorean Theorem (Meaning)

What kind of triangle is the triangle in the original problem?

- How do you know?

- Where would the following numbers belong in the diagram?

$$\{1, 2, 3, \sqrt{2}, \sqrt{3}\}$$

CONVERSATION PATHS (ANNOTATED)*

A. Problem Solving: Orientation

What is this question asking you?

> ***To classify the three angles of a triangle.***

- What makes it challenging?

> ***It might be challenging because there is no visual representation, two side lengths are irrational numbers, and none of the angle measures are provided.***

- Without being precise on the lengths, could you sketch this triangle?

> ***Are students able to translate the problem into a picture?***

- Which angle would be $\angle ABC$ in your diagram?

> ***$\angle\text{ABC}$ is opposite from side AC.***

B. Content: Pythagorean Theorem (Interpretation, Meaning)

Can line segments of length 1 cm, 2 cm, and 3 cm make a triangle?

> ***No. If the 1 cm and 2 cm lengths were placed at the opposite ends of the 3 cm length, they would only meet if they were directly on the 3 cm length. No triangle would be formed.***
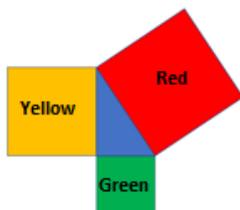
- In order to make a triangle, what must be true about the three line segments?

> ***The sum of the two shorter lengths must be longer than the third.***

- Can line segment lengths 1, √2, and √3 even make a triangle? How do you know?

> ***Yes. The sum of the two shorter lengths (1 + √2 ≈ 2.414) is greater than the length of third side √3 ≈ 1.732.***

What does this figure have to do with the Pythagorean theorem?

> ***The sum of the squares of the lengths of the two shorter sides is equal to the square of the length of the third side. That is, the area of the yellow square plus the area of the green square is equal to the area of the red square.***

- What happens if $c^2 > a^2 + b^2$? How can this diagram help you explain it?

> ***The triangle would have one obtuse angle. The diagram might help by showing that since the area of the red square is greater than the sum of the areas of the yellow and green squares, the angle across from c is greater than 90 degrees.***

- What happens if $c^2 < a^2 + b^2$? How can this diagram help you explain it?

> ***The triangle would have three acute angles. The diagram might help explain by showing that since the area of the red square is less than the sum of the areas of the yellow and green squares, the angle across from c is less than 90 degrees.***

C. Content: Equations (Square roots)

In general, is $\sqrt{(a^2+b^2)}=a+b$? Explain.

$$\sqrt{a^2+b^2} = a+b$$

No, it is not always true. It is true only if either a = 0 or b = 0, or if both a and b equal zero.

- Is $\sqrt{(a^2)}+\sqrt{(b^2)}=a+b$?

$$\sqrt{a^2} + \sqrt{b^2} = a+b$$

Yes, if both a and b are greater than or equal to zero. Otherwise, no.

- Is $\sqrt{(a^2)}+\sqrt{(b^2)}=|a|+|b|$?

$$\sqrt{a^2} + \sqrt{b^2} = |a| + |b|$$

Yes.

D. Pythagorean Theorem (Meaning)

What kind of triangle is the triangle in the original problem?
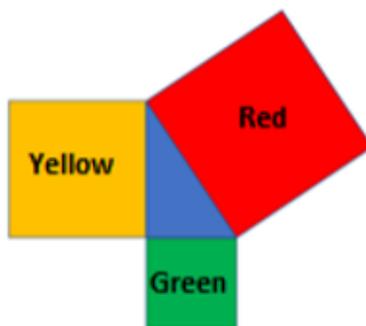
A right triangle.

- How do you know?

The sum of the squares of each side of the triangle is equal to the square of the hypotenuse. The Pythagorean theorem: $1^2+ (\sqrt{2})^2= (\sqrt{3})^2$

- Where would the following numbers belong in the diagram?

Each side of the yellow square has length $\sqrt{2}$. Each side of the green square has length 1. Each side of the red square has length $\sqrt{3}$. The area of the green square is 1 square unit. The area of the yellow square is 2 square units. The area of the red square is 3 square units.

$$\{1, 2, 3, \sqrt{2}, \sqrt{3}\}$$
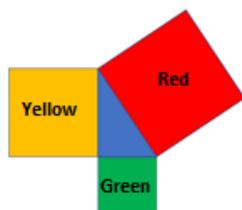
8.4 Shareables*

Use the information to answer the question.

A triangle ABC has side length $|AB| = 1$, $|BC| = \sqrt{2}$, and $|AC| = \sqrt{3}$.

What type of angle is each angle in the triangle? Choose a type for each angle.

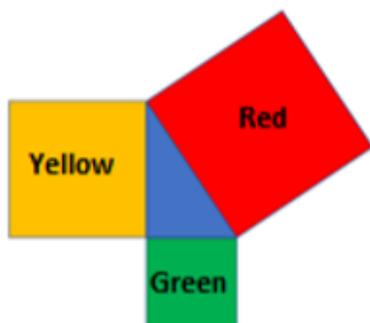| Angle	Type		
$\angle ABC$	acute	obtuse	right
$\angle BCA$	acute	obtuse	right
$\angle CAB$	acute	obtuse	right

B.

C.

$$\sqrt{a^2 + b^2} = a + b$$

$$\sqrt{a^2} + \sqrt{b^2} = a + b$$

$$\sqrt{a^2} + \sqrt{b^2} = |a| + |b|$$

D.

$$\{1, 2, 3, \sqrt{2}, \sqrt{3}\}$$

8.5 The Equal Sign

This activity focuses on student thinking about solving linear equations in one variable.

The Equal Sign: The equal sign works in multiple ways in mathematics, even though the symbol does not change. Students should be able to think about the symbol as being relational (e.g., 2 + 4 = 5 + 1) and as being operational (i.e., the output of a computation), and should be able to determine which role the symbol is playing based on the situation. Keep an eye out for when students may put together equal signs. If this happens, prompting the student with questions about what the equals sign means may help. For example, suppose a student writes: 9 x 8 = 72 = 126 – 72 = 54 (in this case, the student might be using the equal sign to say "next, I will...").

ITEM ALIGNMENT CCSS: 8.EE.C.7b	*This item focuses on solving an equation. However, it also provides an opportunity to talk about strategies, systems, and the meaning of an equation.*

THE CONVERSATION STARTER

Solve for x.

$$17 + 5(2x - 9) = 4 - 2x + 7$$

Enter the answer in the box.

CONVERSATION PATHS (QUESTIONS ONLY)*

A. Problem Solving: Orientation & Strategy

What does it mean to "solve for x"?

- What does the equal sign mean in this context?

- Is it possible for this equation to have exactly two solutions? Three solutions?

- How could you know if your final answer is correct?

- Can you add the 17 and the 5 on the left side? Why or why not?

- What does $5(2x - 9)$ mean?

- Could you subtract 17 from both sides?

- Would that be helpful? Why can you do that?

- What strategies can you use to solve this equation?

B. Content: Equations (Meaning)

How are these alike? How are they different?

$$17 + 5(2x - 9) = 4 - 2x + 7 \qquad \begin{cases} y = 17 + 5(2x - 9) \\ y = 4 - 2x + 7 \end{cases}$$

Could you use a graph to solve the equation in the original problem?

CONVERSATION PATHS (ANNOTATED)*

A. Problem Solving: Orientation & Strategy

What does it mean to "solve for x"?

> *It means to determine the value of the unknown quantity x such that the statement of equality is true.*

- What does the equal sign mean in this context?

  It means that the two expressions on either side of the equal sign have the same value.

- Is it possible for this equation to have exactly two solutions? Three solutions?

  Some equations can have two or three solutions, but this equation could not. One thought could be to imagine each side as the equation of a line. Two lines could only intersect at one point (one solution), never (no solutions), or they could be coincident (the same line). Another approach could be to imagine solving the equation and realizing one could find a single value for x.

- How could you know if your final answer is correct?

  If the final answer is substituted for x in the equation, and the equation is true, then we have found a solution. That is, for that value of x, the value of the expression 17 + 5 (2x – 9) is the same as the value of 4 – 2x + 7.

- Can you add the 17 and the 5 on the left side? Why or why not?

  No, 17 is being added to the 5 copies of (2x – 9), and not to the number 5 itself.

- What does 5(2x – 9) mean?

  It indicates that we have 5 copies of the quantity 2x – 9.
  We could write or think (2x – 9) + (2x – 9) + (2x – 9) + (2x – 9) + (2x – 9), or 10x – 45.

- Could you subtract 17 from both sides?

  Yes, we can subtract any number from both sides, as long as we subtract it from both sides.

- Would that be helpful? Why can you do that?

  It might help, such as if we want all x terms to be on the left side of the equation. The addition property of equality lets us do that.

- What strategies can you use to solve this equation?

  In addition to step-by-step solving, another idea could be to create a graph where each side of the equal sign is a function. The point of intersection would help us to determine the solution to the equation.

B. Content: Equations (Meaning)

How are these alike? How are they different?

> *They are alike in that the same expressions are used in both. Also, in the system on the right, if we substitute the y in the first equation with the y in the second, the result is the equation on the left. They are different in that the one on the left is a single equation involving a single variable. The answer is a single value of x. The set of equations on the right consists of two equations with two variables, so the answer will be an ordered pair, though it will have the same value for x as the equation on the left.*

Could you use a graph to solve the equation in the original problem?

$$17 + 5(2x - 9) = 4 - 2x + 7 \qquad \begin{cases} y = 17 + 5(2x - 9) \\ \quad y = 4 - 2x + 7 \end{cases}$$

> *Yes. If the two equations on the right are graphed, the x-coordinate of the point of intersection is the solution to the equation on the left.*

8.5 Shareables*

Solve for x.

$$17 + 5(2x - 9) = 4 - 2x + 7$$

Enter the answer in the box.

B.

$$17 + 5(2x - 9) = 4 - 2x + 7$$
$$\begin{cases} y = 17 + 5(2x - 9) \\ y = 4 - 2x + 7 \end{cases}$$